

Magnetic QED

C. Ford

School of Mathematical Sciences

University College Dublin

Belfield, Dublin 4

Ireland

Christopher.Ford@ucd.ie

Abstract

A non-Hermitian form of QED is presented which describes interacting Dirac monopoles. The theory is related by a canonical transformation to a model proposed by Milton. As in Hermitian QED an abelian gauge potential is coupled to a four-component fermion. Under proper Lorentz transformations and time-reversal the fermion field transforms like a Dirac spinor but has a non-standard parity transformation. This implements the property that magnetic charge, unlike electric charge, is parity-odd. A consequence of the non-Hermiticity is that there is an attractive force between identical charged particles, at least in the weakly coupled regime. This effect can be understood even at the classical level; a simple calculation of the force between classical Dirac monopoles is presented which shows that like charge monopoles attract and opposite charges repel.

This paper concerns the physical interpretation of non-Hermitian forms of quantum electrodynamics (QED). In non-relativistic quantum mechanics some very simple non-Hermitian Hamiltonians have been shown to exhibit a positive spectrum and unitary time-evolution [1, 2]. These remarkable properties have also been identified in certain quantum field theories [3, 4, 5]. Tentative steps have been taken to apply these ideas to gauge theory. In particular, Milton [6] has proposed a non-Hermitian version of QED. Unlike standard QED, parity \mathcal{P} and time-reversal \mathcal{T} are not symmetries of the theory. However, the combined symmetry \mathcal{PT} is respected and on this basis the theory is expected to exhibit a real energy spectrum and unitary time-evolution. The theory is also asymptotically free.

In this paper an alternative Lagrangian for non-Hermitian QED is given. This theory is symmetric under \mathcal{P} and \mathcal{T} . Under parity and time-reversal the field strength transforms like the Maxwell dual field strength. This, together with the transformation properties of the current, suggests that the theory is a magnetic form of QED; the elementary fermions carry magnetic rather than electric charge. As the magnetic current is anti-Hermitian the force between like charge particles is *attractive* rather than repulsive. In fact, it is possible to understand this effect even at the classical level. A simple, albeit formal, argument is given that shows that the force between two like charge Dirac monopoles is indeed attractive. Opposite charges repel. A construction of free-field representations for the gauge and matter fields is outlined. This analysis shows that the new theory is related by a canonical transformation to that of Milton.

Massless QED is based on the Lagrangian (the metric is $\text{diag}(1, -1, -1, -1)$)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu\partial_\mu\psi + e\bar{\psi}\gamma^\mu A_\mu\psi, \quad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Here A_μ is a $U(1)$ gauge potential and ψ is a Dirac spinor. The corresponding quantum theory has a Hermitian Hamiltonian and is symmetric under parity \mathcal{P} and time-reversal \mathcal{T} . Milton considered the Lagrangian [6]¹

$$\mathcal{L} = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + i\bar{\psi}\gamma^\mu\partial_\mu\psi + ig\bar{\psi}\gamma^\mu B_\mu\psi, \quad (2)$$

with $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, B_μ being an abelian gauge potential, ψ a Dirac spinor and g a real coupling constant. The theory couples a gauge potential B_μ (assumed to be Hermitian) to the anti-Hermitian current

$$j^\mu = ig\bar{\psi}\gamma^\mu\psi. \quad (3)$$

A consequence of the anti-hermiticity is that the current has a non-standard transformation law under the (anti-unitary) operation of time-reversal

$$\mathcal{T}j^0(\mathbf{r}, t)\mathcal{T}^{-1} = -j^0(\mathbf{r}, -t), \quad \mathcal{T}\mathbf{j}(\mathbf{r}, t)\mathcal{T}^{-1} = \mathbf{j}(\mathbf{r}, -t). \quad (4)$$

Assuming the gauge potential transforms in the usual way under time reversal, ie.

$$\mathcal{T}B_0(\mathbf{r}, t)\mathcal{T}^{-1} = B_0(\mathbf{r}, -t), \quad \mathcal{T}\mathbf{B}(\mathbf{r}, t)\mathcal{T}^{-1} = -\mathbf{B}(\mathbf{r}, -t), \quad (5)$$

¹ In [6] a real representation for Dirac spinors was adopted. In this paper, as also in [7], a conventional complex representation is assumed.

the theory is not symmetric with respect to time-reversal. The author of [6] was, however, seeking a \mathcal{PT} -symmetric theory. To achieve this the gauge potential B_μ is required to have the pseudovector parity transformation

$$\mathcal{P}B_0(\mathbf{r}, t)\mathcal{P}^{-1} = -B_0(-\mathbf{r}, t), \quad \mathcal{P}\mathbf{B}(\mathbf{r}, t)\mathcal{P}^{-1} = \mathbf{B}(-\mathbf{r}, t). \quad (6)$$

The resulting theory is not parity-symmetric but \mathcal{PT} is a symmetry of the theory.

Now consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}H_{\mu\nu}H^{\mu\nu} + i\bar{\lambda}\gamma^\mu\partial_\mu\lambda + ig\bar{\lambda}\gamma^\mu V_\mu\lambda. \quad (7)$$

Here $H_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ where the gauge potential V_μ has unconventional transformations under both \mathcal{P} and \mathcal{T} , that is

$$\mathcal{T}V_0(\mathbf{r}, t)\mathcal{T}^{-1} = -V_0(\mathbf{r}, -t), \quad \mathcal{T}\mathbf{V}(\mathbf{r}, t)\mathcal{T}^{-1} = \mathbf{V}(\mathbf{r}, -t), \quad (8)$$

and

$$\mathcal{P}V_0(\mathbf{r}, t)\mathcal{P}^{-1} = -V_0(-\mathbf{r}, t), \quad \mathcal{P}\mathbf{V}(\mathbf{r}, t)\mathcal{P}^{-1} = \mathbf{V}(-\mathbf{r}, t). \quad (9)$$

The spinor field λ transforms like a Dirac spinor under proper Lorentz transformations and time-reversal. Under parity it transforms as

$$\mathcal{P}\lambda_\alpha(\mathbf{r}, t)\mathcal{P}^{-1} = P_{\alpha\beta}\lambda_\beta^\dagger(-\mathbf{r}, t), \quad (10)$$

where $P_{\alpha\beta}$ denotes the matrix elements of the Dirac matrix $i\gamma^0\gamma^2$ (here it is assumed that $\gamma_0 = \gamma_0^T$ and $\gamma_2 = \gamma_2^T$). This does not look like a parity transformation; it is actually the standard form of the \mathcal{CP} transformation for Dirac spinor fields. In much the same way that \mathcal{CP} is unitary in standard QED (even though in one-particle Dirac theory it is anti-unitary) the above \mathcal{P} transformation is unitary. The theory couples the Hermitian gauge potential V_μ to the anti-Hermitian current

$$j^\mu = ig\bar{\lambda}\gamma^\mu\lambda. \quad (11)$$

Under \mathcal{P} and \mathcal{T}^2

$$\mathcal{T}^{-1}j^0(\mathbf{r}, t)\mathcal{T} = -j^0(\mathbf{r}, -t), \quad \mathcal{P}^{-1}\mathbf{j}(\mathbf{r}, t)\mathcal{P} = \mathbf{j}(-\mathbf{r}, t). \quad (12)$$

²Another anti-Hermitian current [8] with exactly these transformation properties is $j^\mu = ig\bar{\psi}\gamma^\mu\gamma^5\psi$. However, as charge conservation is spoiled by the chiral anomaly it is not clear whether it leads to a consistent quantum field theory.

This non-Hermitian theory is symmetric under \mathcal{T} and \mathcal{P} ; the non-standard transformation properties of V_μ compensate for those of j^μ . The field strength $H_{\mu\nu}$ transforms like the Maxwell dual field strength and satisfies the \mathcal{P} and \mathcal{T} symmetric equation of motion

$$\partial_\mu H^{\mu\nu} = j^\nu. \quad (13)$$

This indicates that the theory is a magnetic version of QED. That magnetic charge is parity odd just as electric charge is \mathcal{CP} -odd ‘explains’ the \mathcal{CP} -like form of the parity transformation for V and λ .

It remains to interpret the non-Hermiticity of the theory. A consequence of the anti-hermiticity of the current is an attractive force between identical charged particles, at least in the weak coupling regime. In fact, this effect can be understood for *classical* Dirac monopoles. In the absence of a Lorentz force formula for magnetic charges it is proposed to compute the force between two static monopoles using the electromagnetic energy density formula

$$u = \frac{1}{2}(E^2 + B^2). \quad (14)$$

As the electric and magnetic fields enter this formula symmetrically one might expect that the force between Dirac monopoles follows exactly the same pattern as for electric charges. It is argued below that the Dirac string of the monopole breaks this symmetry leading to an attractive force for like charges.

A stationary Dirac monopole [9] centered at the origin can be described by the vector potential

$$\mathbf{A} = \frac{g}{4\pi r} \frac{y\mathbf{i} - x\mathbf{j}}{(r - z)}, \quad (15)$$

where g is the magnetic charge. In this gauge the Dirac string lies on the positive z -axis. The magnetic field is

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{g\mathbf{r}}{4\pi r^3} - g\theta(z)\delta(y)\delta(z)\mathbf{k}. \quad (16)$$

Consider two Dirac monopoles with magnetic charge g centred at the points $\mathbf{r} = a\mathbf{k}$ and $\mathbf{r} = -a\mathbf{k}$, respectively. The total magnetic energy can be expressed formally as the integral

$$E = \frac{1}{2} \int d^3x (\mathbf{B}_1 \cdot \mathbf{B}_1 + \mathbf{B}_2 \cdot \mathbf{B}_2 + 2\mathbf{B}_1 \cdot \mathbf{B}_2), \quad (17)$$

where \mathbf{B}_1 and \mathbf{B}_2 are the contributions to the magnetic field due to the first and second monopole, respectively. Now

$$U = \int d^3x \mathbf{B}_1 \cdot \mathbf{B}_2 \quad (18)$$

is the part of the energy needed for the force computation since the remainder comprises the infinite self energies of the two monopoles which do not depend on the monopole separation. Inserting the two magnetic fields into (18) gives

$$U = \frac{g^2}{16\pi^2} \int d^3x \left(\frac{\mathbf{r} - a\mathbf{k}}{|\mathbf{r} - a\mathbf{k}|^3} \cdot \frac{\mathbf{r} + a\mathbf{k}}{|\mathbf{r} + a\mathbf{k}|^3} - 4\pi\theta(z - a)\delta(x)\delta(y)\mathbf{k} \cdot \frac{\mathbf{r} + a\mathbf{k}}{|\mathbf{r} + a\mathbf{k}|^3} + 4\pi\frac{\mathbf{r} - a\mathbf{k}}{|\mathbf{r} - a\mathbf{k}|^3} \cdot \theta(-z - a)\delta(x)\delta(y)\mathbf{k} \right). \quad (19)$$

Here \mathbf{B}_1 is a translation of (16) and

$$\mathbf{B}_2 = \frac{g}{4\pi} \frac{\mathbf{r} + a\mathbf{k}}{|\mathbf{r} + a\mathbf{k}|^3} + g\theta(-z - a)\delta(y)\delta(z)\mathbf{k}, \quad (20)$$

so that the Dirac string of the second monopole lies on the part of the negative axis below $z = -a$. The first term in (19) gives the expected Coulomb repulsion, $g^2/(8\pi|a|)$. Performing the other two integrals gives a contribution twice the Coulomb term but with the opposite sign. Accordingly,

$$U = -\frac{g^2}{8\pi|a|}, \quad (21)$$

giving an attractive force between like charge monopoles. Similarly, the force between opposite charges is repulsive. In this computation we have taken the Dirac strings to lie on the z -axis. However, the result is independent of the string placement (provided the two strings do not intersect which would lead to an ill-defined cross term in the U integral). A relativistic force law that incorporates the magnetic attraction property is

$$m \frac{d^2 x_\mu}{d\tau^2} = (eF_{\mu\nu} - g\tilde{F}_{\mu\nu}) \frac{dx^\nu}{d\tau}, \quad (22)$$

where τ denotes the proper time and $\tilde{F}_{\mu\nu}$ is the Hodge dual of $F_{\mu\nu}$. Maxwell's equations are

$$\partial_\mu F^{\mu\nu} = j_e^\nu, \quad \partial_\mu \tilde{F}^{\mu\nu} = j_m^\nu, \quad (23)$$

where j_e^μ and j_m^μ are the electric and magnetic currents, respectively.

To conclude, the derivation of free-field representations for the QED theories is outlined. It is instructive to start with the photon field for ordinary QED (see for example [10]). A free photon field (the gauge is fixed so that $A_0 = 0$ and $\nabla \cdot \mathbf{A}$) takes the form

$$\mathbf{A}(\mathbf{r}, t) = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_{\lambda=1}^2 \mathbf{e}(k, \lambda) [a(k, \lambda)e^{-ik \cdot x} + a^\dagger(k, \lambda)e^{ik \cdot x}], \quad (24)$$

Here $k_0 = \omega = |\mathbf{k}|$ so that $k_\mu k^\mu = 0$. The polarization vectors, $\mathbf{e}(k, \lambda)$ $\lambda = 1, 2$, are orthogonal to \mathbf{k} , and satisfy

$$\mathbf{e}(k, \lambda) \cdot \mathbf{e}(k', \lambda) = \delta_{\lambda\lambda'}, \quad \mathbf{e}(-k, 1) = -\mathbf{e}(k, 1), \quad \mathbf{e}(-k, 2) = +\mathbf{e}(k, 2). \quad (25)$$

The creation and annihilation operators obey the commutation relations

$$[a(k, \lambda), a^\dagger(k', \lambda')] = \delta^3(\mathbf{k} - \mathbf{k}')\delta_{\lambda\lambda'}, \quad (26)$$

and

$$[a(k, \lambda), a(k', \lambda')] = [a^\dagger(k, \lambda), a^\dagger(k', \lambda')] = 0. \quad (27)$$

The action of the discrete symmetries is as follows

$$\mathcal{T}^{-1}\mathbf{A}(\mathbf{r}, t)\mathcal{T} = -\mathbf{A}(\mathbf{r}, -t), \quad \mathcal{P}^{-1}\mathbf{A}(\mathbf{r}, t)\mathcal{P} = -\mathbf{A}(-\mathbf{r}, t), \quad (\mathcal{CP})^{-1}\mathbf{A}(\mathbf{r}, t)\mathcal{CP} = \mathbf{A}(-\mathbf{r}, t). \quad (28)$$

\mathcal{P} and \mathcal{CP} have the representations

$$\mathcal{P} = \exp \left[-\frac{i\pi}{2} \int d^3k \sum_{\lambda=1}^2 \left(a^\dagger(k, \lambda)a(k, \lambda) + a^\dagger(k, \lambda)a(-k, \lambda) \right) \right], \quad (29)$$

and

$$\mathcal{CP} = \exp \left[\frac{i\pi}{2} \int d^3k \sum_{\lambda=1}^2 \left(a^\dagger(k, \lambda)a(k, \lambda) - a^\dagger(k, \lambda)a(-k, \lambda) \right) \right]. \quad (30)$$

We require a Hermitian quantum field V_μ satisfying the same commutation relations as A_μ but with the *opposite* transformations to A_μ under \mathcal{T} , \mathcal{P} and \mathcal{CP} . Consider

$$\mathbf{V}(\mathbf{r}, t) = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_{\lambda=1}^2 \mathbf{e}(k, \lambda) \left[ia(k, \lambda)e^{-ik \cdot x} - ia^\dagger(k, \lambda)e^{ik \cdot x} \right]. \quad (31)$$

This is the same as $\mathbf{A}(\mathbf{r}, t)$ but with $a(k, \lambda)$ replaced by $ia(k, \lambda)$ and $a^\dagger(k, \lambda)$ replaced by $-ia^\dagger(k, \lambda)$, a canonical transformation. The i insertions switch the time-reversal properties of the field so that (8) holds. Note that \mathcal{P} and \mathcal{CP} are unaffected so that (9) is not satisfied. However, all one needs to do is to swap \mathcal{P} and \mathcal{CP} . That is the parity operator \mathcal{P} is *defined* to be the standard form of \mathcal{CP} and \mathcal{CP} is defined to be the standard form of \mathcal{P} . The same procedure yields a free-field representation of the fermion λ ; simply take a standard Dirac fermion and exchange the definitions of \mathcal{P} and \mathcal{CP} . This provides a fermion field λ with a standard \mathcal{T} transform and the non-standard parity transformation (10). Interacting

fields may be formally defined in the usual way through $V_\mu^{int}(\mathbf{r}, t) = e^{iHt}V_\mu(\mathbf{r}, 0)e^{-iHt}$ and $\lambda^{int}(\mathbf{r}, t) = e^{iHt}\lambda(\mathbf{r}, 0)e^{-iHt}$.

To define B_μ in the Milton theory take B_μ to be A_μ and swap the definition of \mathcal{P} and \mathcal{CP} . For the fermion field the standard definitions of \mathcal{P} and \mathcal{CP} are retained. This is consistent since for free fields the parity operator decomposes into commuting gauge and fermionic pieces, $\mathcal{P} = \mathcal{P}_{gauge}\mathcal{P}_{fermion}$; one can choose a non-standard ‘magnetic’ \mathcal{P}_{gauge} together with a standard ‘electric’ $\mathcal{P}_{fermion}$. In fact, taking the ‘non-standard’ form for both \mathcal{P}_{gauge} and $\mathcal{P}_{fermion}$ gives a parity-symmetric theory. Then the fermion would transform like λ and one can write the Lagrangian as

$$\mathcal{L} = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + i\bar{\lambda}\gamma^\mu\partial_\mu\lambda + ig\bar{\lambda}\gamma^\mu B_\mu\lambda, \quad (32)$$

which is (7) with V_μ replaced by B_μ . As B_μ and V_μ are related by a canonical transformation so are the two non-Hermitian theories.

References

- [1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998)
[arXiv:physics/9712001].
- [2] P. Dorey, C. Dunning and R. Tateo, J. Phys. A **34**, 5679 (2001)
[arXiv:hep-th/0103051].
- [3] F. Kleefeld, ‘Non-Hermitian quantum theory and its holomorphic representation: Introduction and some Applications’, [arXiv:hep-th/0408028].
- [4] F. Kleefeld, ‘Non-Hermitian quantum theory and its holomorphic representation: Introduction and applications’, [arXiv:hep-th/0408097].
- [5] C. M. Bender, H. F. Jones and R. J. Rivers, Phys. Lett. B **625**, 333 (2005)
[arXiv:hep-th/0508105].
- [6] K. A. Milton, Czech. J. Phys. **54**, 85 (2004) [arXiv:hep-th/0308035].
- [7] C. M. Bender, I. Cavero-Pelaez, K. A. Milton and K. V. Shajesh, Phys. Lett. B **613**, 97 (2005) [arXiv:hep-th/0501180].

- [8] C. M. Bender and K. A. Milton, J. Phys. A **32**, L87 (1999).
- [9] P. A. M. Dirac, Proc. Royal Society London A133 (1931) 60.
- [10] J. D. Bjorken and S. Drell, Relativistic Quantum Fields, McGraw-Hill 1965.